A QUEUEING APPROACH TO ALLOCATING SPARES OF RECOVERABLE ITEMS AMONG ADVANCE NAVAL BASES

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THESIS

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TO ALLOCATING SPARES OF RECOVERABLE ITEMS
AMONG ADVANCE NAVAL BASES

by

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number of back orders for the total system are derived. A marginal approach is proposed for determining the optimal allocation of spares to each advance base.

A Queueing Approach
to Allocating Spares of Recoverable Items
Among Advance Naval Bases

by

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I. INTRODUCTION

Large organizations with operating subsidiaries, located at distances from one another are often faced with problems of shortage of items in use. The shortage may affect the functioning of a subsidiary, and in turn, may affect the performance of the organization as a whole. An organization often cannot avoid shortages unless it can stock an infinite or very large number of spares, and carrying a large number of spares of an item to reduce shortages is sometimes neither practical nor economical. In most cases there are limitations on the number of spares that can be held. This may occur from the limitations of budget, storage space available, or delays in resupply. Also, in some cases an organization may limit itself to a certain number of spares of some specific items in order to avoid great losses when technology changes, and items in hand become obsolete. Only in a few cases where shortage of an item is considered catastrophic will spares be stocked at higher levels. So in general, no matter what the limitations are from other factors or from the purpose of the organization itself, spares for an item will be held only to a limited number. This will allow some shortages of items in use to occur among the subsidiaries at any time. Shortages such as these are sometimes called back orders when needed items will be eventually supplied. Given a certain number of spares that an organization can hold, the

back orders might depend upon the allocation of those spares as well as other factors.

Our interest is in reducing the amount of back orders to a minimum when a limited amount of spare items are available in the organization. This can be done by finding an optimum allocation of those spares among the operating units. To make the problem clear, a Navy organization will be used as a sample organization to be solved for optimum solutions. Other organizations can use the solution derived in this paper, as well.

In this paper the problem involves an item which is used at many distant subsidiaries (advance bases, in case of the Navy). The item is recoverable and can be repaired within the area in which it is used. Items such as engines and pumps are of this type.

In the work which follows, parts of the problem will be considered as queueing systems and the whole problem viewed as an allocation problem. There are five chapters in this paper. Chapter 2 is concerned with describing the problem, defining variables, and stating assumptions. In Chapter 3, two models are derived, one for the case when the number of items in use is large, and the other for the case when the number in use is small. Measures of effectiveness are established in Chapter 4 and a numerical example is presented. The last chapter provides a conclusion of the work and gives suggestion for further work on problems of this type.

II. DESCRIPTION OF PROBLEM

It is necessary to understand the problem clearly before we start to solve it. In this chapter we will discuss the nature of the problem and how it arises. Following this, a measure of effectiveness will be proposed to use as a criterion for the model developed in a later chapter.

Consider a Navy with assigned tasks for defense operations at sea involving ship deployments in areas distant from home ports and supply depots. This usually leads to the establishment of a number of advance bases to support these ships at their operating areas. One mission of advance bases is supply and most of the advance bases will usually have some spare items on stock as well as limited repair facilities to service certain recoverable items. Ordinarily, there are spare items onboard ships too.

Our interest in this paper is in recoverable items. Usually, when an item on a ship fails, it will be replaced by a spare item from ship's stock if one is available; otherwise a spare item will be requested from the advance base. The failed item will be sent to the repair facility at the base for repair. (A failed item that receives minor repair on ship is out of our interest here since this usually takes a small amount of time and only slightly effects the performance of ship.) Ship's inventory will be replenished by a spare item from the advance base and when the item is repaired

at the repair facility, it will enter the advance base's inventory.

This system permits a ship to maintain its performance as long as there are spare items on the ship or at the advance base. When the advance base and all ships in that base's area run out of spare stocks, however, a failure of an item on the ship may cause a deterioration of the ship's performance during the time the item is back ordered, waiting for the repair facility to replace the item. As we shall use the term, 'back orders' means the number of items that is shorted at any time among ships at an advance base.

A problem of particular interest here is that of specifying how many spare items to allocate to an advance base and its operating units, since there are limitations on total number of items that the Navy can share among these advance bases due to limitations on budget, etc. Giving more to one advance base means giving less to another, and may influence total back orders throughout the fleet.

If we consider a specific item, it is likely that certain information will be available. We should be able to determine the total number of items in use by the operating units supported by each advance base and the total number of spares in the Navy available to allocate to advance bases and ships supported by those bases. We will assume that the failure rate of an item on an operating unit at any advance base can be predicted from past data or from the task assigned. The mean repair rate for the item at each base's repair facility should be estimatable from old records.

Since ships operate in areas close to the advance base to which they are assigned, we can consider the ordering and shipping time for spares from that base to a ship as negligible. Also, since spares on one ship can be transferred to another ship in the same operating area, we shall consider the spares on ships as if they were in stock at the advance base, forming one inventory of spares for the advance base and its ships. Hence, the term "spare at advance base" will include the spares on ships assigned to that base as well.

A. CRITERION

We now consider the measure of effectiveness (MOE) for this allocation problem. If we think of the Navy as a whole, the total cost for spares of all items seems to be important, although this will probably be determined by budget. Allocation of budget among items is important, but our problem is at a different level. We are interested in a quantity of a single item which the Navy must allocate among the advance bases. Our MOE need not depend directly on cost, and one reasonable MOE for our problem would be the number of back orders that ships or operating units incur over a given time period. Since failures occur randomly, the number of back orders at a given time is a random variable. Using expected back orders as an MOE, our problem is one of allocating spares of a specific item to all bases in order to minimize the expected back orders.

The claim that expected back orders is a reasonable MOE implies that failures are essentially equally important, or, more basically, that every ship at every base has an equally important task so that the effects of a shortage on ships at one base are the same as on ships at other bases.

B. THE APPROACH OF THIS THESIS

In the chapters which follow we will develop an analytic model portraying expected back orders at an advance base in terms of the allocation of spare items among all advance bases. Then we shall propose an optimizing procedure permitting determination of the allocation of spares to minimize expected back orders. It should be noted that our work is directed toward an item which is always repairable when it fails, and which always may be repaired by the repair facility at the ship's advance base. Items may not be transferred among advance bases.

To increase our understanding of the problem, Figure 1 represents a part of the system (one advance base) and we see that failed items on ships will be replaced quickly as long as spare items (including repaired items) do not run out of stock. When spare items run out (and no repaired items are left) then failed items on ships will not be replaced and back orders will occur.

This system of a repair facility with items on ships and at the repair facility can be considered as a queueing system with failed items as customers and the repair facility as a server. Then we can show our system more precisely as in Figure 2.

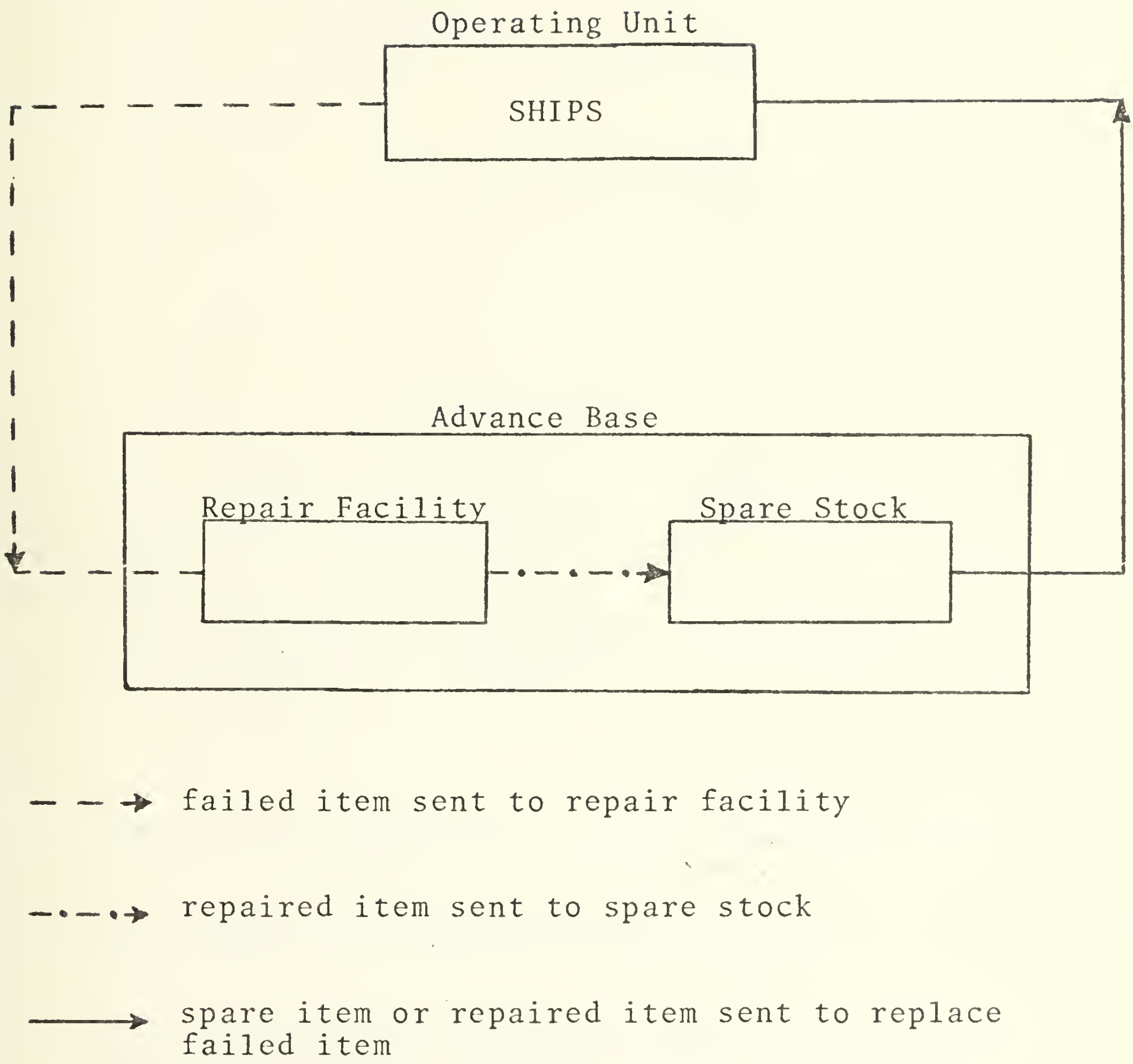

Figure 1. System at an Advance Base.

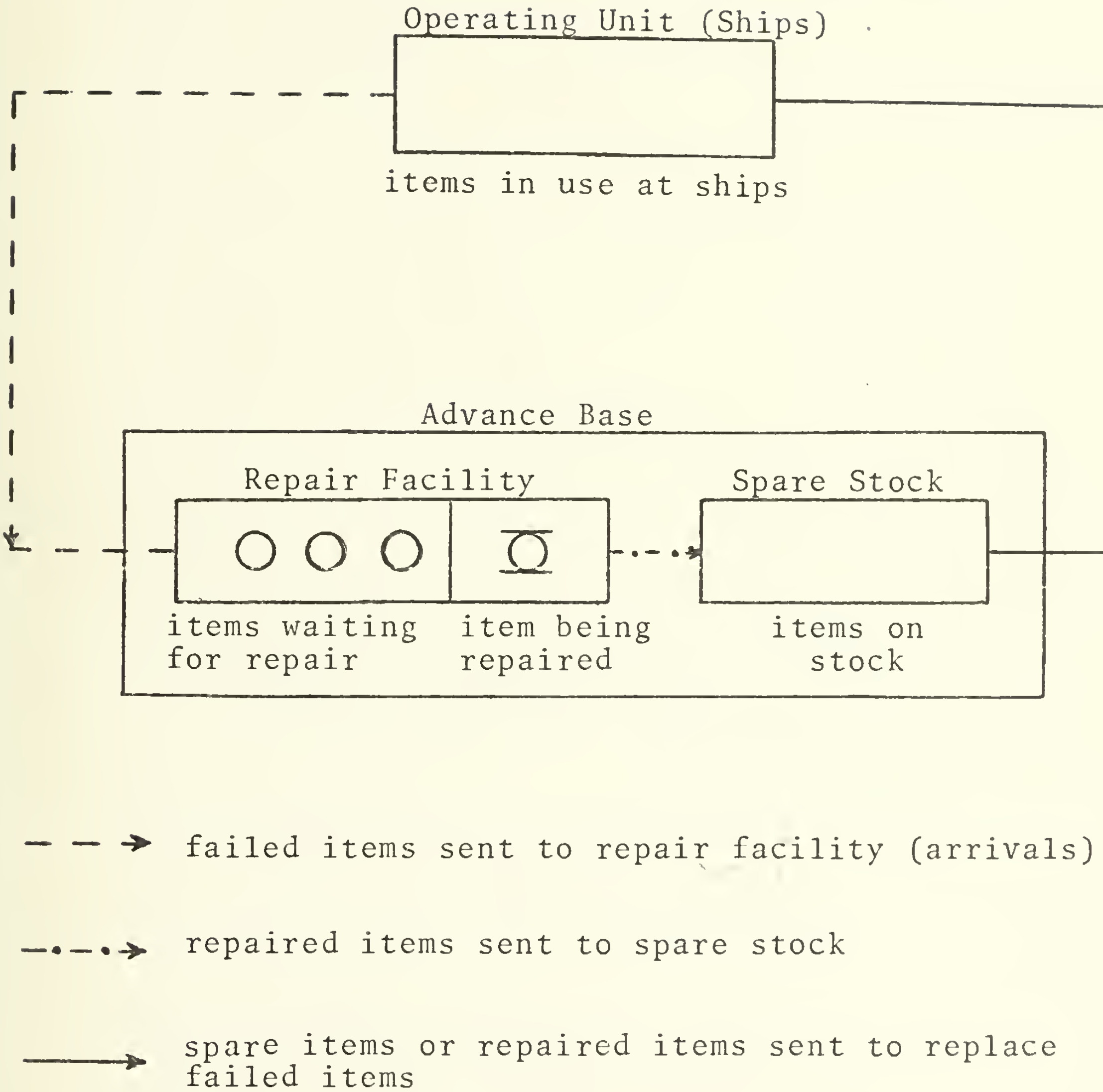

Figure 2. Queueing System at One Base's Repair Facility.

This structure permits the number of items delayed at the repair facility and the number of items in stock to be determined. This, in turn, allows computation of the number of items still in use on ships and the back orders incurred. All of this will be developed in detail in the next chapter.

III. MODELING THE ADVANCE BASE REPAIR SYSTEM

This chapter is concerned with constructing a model of the advance base repair system to assist in allocating spares among advance bases. After the variables to be used are defined and the assumptions are stated, we will consider models for two cases: an infinite, and a finite number of items in use. The model will be developed in terms of expected back orders.

A. VARIABLES

Variables of concern in the work to follow are defined below. For any advance base,

N = number of items in use,

A_t = number of items at the advance base at time t; this includes items stocked at the base and on ships, items being repaired, and items waiting for repair,

n = number of items at the repair facility; this includes items being repaired and items waiting to be repaired (in the queue),

λ = failure rate of items in use on ships,

μ = repair rate (for failed items) at the base's repair facility,

B.O. = total back order incurred at the ships at an advance base at any time,

E(B.O.) = expected back order incurred at ships in an advance base (as a whole) at any time.

Subscripts will be used to indicate the base under consideration, e.g., N_j = number of item in use (all ships) at j^{th} base.

B. QUEUEING APPROACH

In addition to the discussion of Chapter 2, other assumptions should be stated here. Part of each advance base is viewed as a queueing system with failed items from ships as customers and base's repair facility as the server. It is assumed that customers arrive with a Poisson arrival rate, and that the server serves with exponential service times. Since the problem concerns items of the same kind, there is no priority among them other than the ordinary discipline, 'first in, first out'.

C. THE MODELS

In a birth-death process where

λ_n = rate of birth when there are n items in the system,

μ_n = rate of death when there are n items in the system, and

P_n = probability that there are n items in the system in the long run (steady state),

one can show that [1]

$$P_n = \frac{\lambda_o \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n} P_o ,$$

and

$$P_o = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_o \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n}} .$$

Thus we have

$$P_n = \frac{\Pi_n}{1 + \sum_{n=1}^{\infty} \Pi_n} ,$$

where

$$\Pi_n = \frac{\lambda_o \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n} . \tag{1}$$

This result will be used in the work that follows.

Case 1, N Large

If the number of items in use on the ships attached to an advance base is large so that we can consider N approximately to be infinity, then the model is simply derived. Since $N \approx \infty$, the mean rate of failed items from ships sent to the repair facility can be considered as a constant, i.e., $\lambda_n = \lambda$ (if λ = failure rate of an item in use). This is the "birth rate" to the system at the base's repair facility. Since the rate of repairing ("death rate") should not depend on the number of failed items in the system, μ_n is equal to μ, a constant. Accordingly, we have

$$\Pi_n = \frac{\lambda \cdot \lambda \cdots \lambda}{\mu \cdot \mu \cdots \mu} = \left(\frac{\lambda}{\mu} \right)^n ,$$

and

$$\sum_{n=1}^{\infty} \Pi_n = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^n = \frac{1}{1 - \frac{\lambda}{\mu}} - 1 .$$

However,

$$1 + \sum_{n=1}^{\infty} \Pi_n = \frac{1}{1 - \frac{\lambda}{\mu}} ,$$

and therefore

$$\begin{aligned} P_n &= \left(\frac{\lambda}{\mu} \right)^n \left(1 - \frac{\lambda}{\mu} \right) \\ &= \rho^n (1 - \rho), \end{aligned} \tag{2}$$

where $\rho = \frac{\lambda}{\mu}$.

This equation can be used for each advance base.

We now turn our attention to back orders. Figure 3 shows how items on ships, in the queueing system (waiting for repair and being repaired), and in stock relate to each other.

Relationships between items at different places and the number of back orders are as follows:

n	Stock	A_t	Number of Back Orders (B.O.)
0	A_o	A_o	0
1	A_o-1	A_o	0
2	A_o-2	A_o	0
.	.	.	.
.	.	.	.
.	.	.	.
A_o-1	1	A_o	0
A_o	0	A_o	0
A_o+1	0	A_o+1	1
A_o+2	0	A_o+2	2
.	.	.	.
.	.	.	.
.	.	.	.

Accordingly,

$$B.O. = \begin{cases} 0 & , 0 \le n \le A_o \\ n-A_o & , A_o < n \le A_o + N. \end{cases}$$

Note that A_t is a random variable. In general, it equals A_o at time t = 0 (or in other words, at the starting of the problem). But it may also equal A_o at any time when back

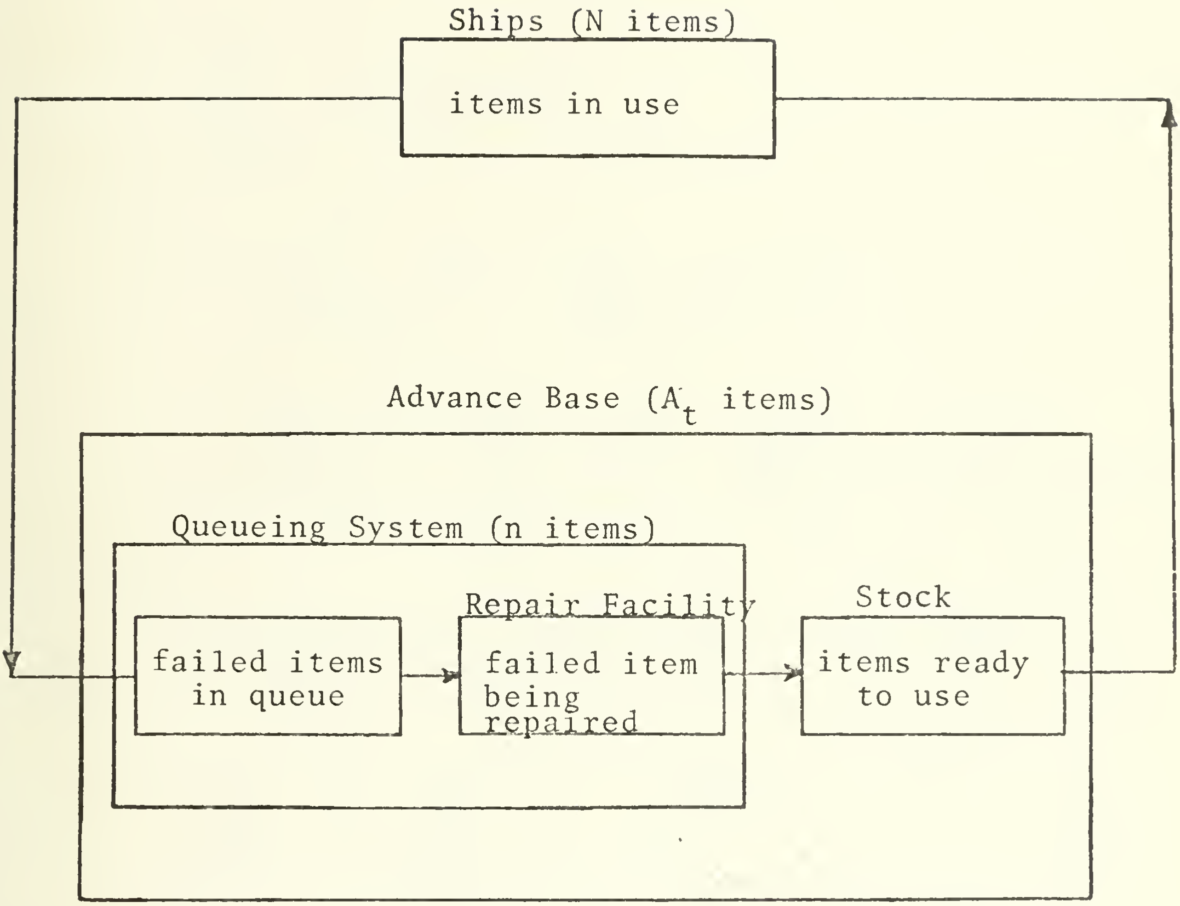

Figure 3. Advance Base Repair System Showing Inventory Levels.

orders are not present and no failed items are in the queueing system.

Expected back orders can be found as follows:

$$E[B.O.] = \sum_{n} (B.O.)P_n$$

$$= \sum_{n=A_o+1}^{\infty} (n-A_o)(1-\rho)\rho^n$$

$$= (1-\rho)\sum_{n=A_o+1}^{\infty} n\rho^n - A_o \sum_{n=A_o+1}^{\infty} (1-\rho)\rho^n$$

$$= (1-\rho)\left[\sum_{n=0}^{\infty} n\rho^n - \sum_{n=0}^{A_o} n\rho^n\right] - A_o\left[\sum_{n=0}^{\infty} (1-\rho)\rho^n - \sum_{n=0}^{A_o} (1-\rho)\rho^n\right]$$

$$= (1-\rho)\left[\frac{\rho}{(1-\rho)^2} - \sum_{n=0}^{A_o} n\rho^n\right] - A_o\left[1 - (1-\rho)\cdot\frac{1-\rho^{A_o+1}}{1-\rho}\right]$$

$$= \frac{\rho}{1-\rho} - (1-\rho)\sum_{n=0}^{A_o} n\rho^n - A_o\rho^{A_o+1}. \qquad (3)$$

Now

$$\sum_{n=0}^{A_o} n\rho^n = 0 + \rho + 2\rho^2 + \cdots + A_o\rho^{A_o},$$

and thus

$$\rho\sum_{n=0}^{A_o} n\rho^n = 0 + \rho^2 + 2\rho^3 + \cdots + A_o\rho^{A_o+1},$$

therefore

$$(1-\rho)\sum_{n=0}^{A_o} n\rho^n = \left(\sum_{n=0}^{A_o} \rho^n - 1\right) - A_o\rho^{A_o+1}$$

$$= \frac{1 - \rho^{A_o+1}}{1 - \rho} - 1 - A_o\rho^{A_o+1} .$$

Substituting this into (3), we have

$$E(B.O.) = \frac{\rho^{A_o+1}}{1 - \rho} = \left(\frac{\lambda}{\mu}\right)^{A_o+1} \left(1 - \frac{\lambda}{\mu}\right)^{-1} . \qquad (4)$$

This is the general form for every advance base.

Case 2, N Finite

When the number of item in use on ships cannot be considered infinite, the rate at which failed items arrive at the base's repair facility will depend upon the number of items left on ships. The rates of arrival are shown below.

n	Stock	A_t	B.O.	Items Left on Ships	Arrival (Failure) Rate, λ_n
0	A_o	A_o	0	N	$N\lambda$
1	A_o-1	A_o	0	N	$N\lambda$
2	A_o-2	A_o	0	N	$N\lambda$
⋮	⋮	⋮	⋮	⋮	⋮
A_o-1	1	A_o	0	N	$N\lambda$
A_o	0	A_o	0	N	$N\lambda$
A_o+1	0	A_o+1	1	N-1	$(N-1)\lambda$
A_o+2	0	A_o+2	2	N-2	$(N-2)\lambda$
⋮	⋮	⋮	⋮	⋮	⋮
A_o+N-1	0	A_o+N-1	N-1	1	λ
A_o+N	0	A_o+N	N	0	0

From the above, one can see that back orders,

$$\text{B.O.} = \begin{cases} 0 & , 0 \le n \le A_o \\ n-A_o & , A_o < n \le A_o+N . \end{cases}$$

The failure rate λ_n stays constant at value $N\lambda$ as long as the number of items in use on the ships is N. When n increases and is greater than A_o, items in use on ships will decrease below N and back orders will occur. This will result in a change in the failure rate:

$$\lambda_n = \begin{cases} N\lambda & , 0 \le n \le A_o \\ (N+A_o-n)\lambda & , A_o < n \le A_o+N . \end{cases}$$

Since the repair rate is $\mu_n = \mu$, we have for $0 \le n \le A_o$,

$$\Pi_n = \frac{\lambda_o \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n} = \frac{N\lambda \cdot N\lambda \cdots N\lambda}{\mu \cdot \mu \cdots \mu} = (N\rho)^n ,$$

for $A_o < n \le A_o+N$,

$$\Pi_n = \left(\frac{N\lambda}{\mu} \right)^{A_o} \frac{N\lambda \cdot (N-1)\lambda \cdots (N+A_o-n+1)\lambda}{\mu \cdot \mu \cdots \mu}$$

$$= \left(\frac{\lambda}{\mu} \right)^n \frac{N^{A_o} N!}{(N+A_o-n)!}$$

$$= \frac{\rho^n N^{A_o} N!}{(N+A_o-n)!} ,$$

and for $n > A_o+N$,

$$\Pi_n = 0. \tag{5}$$

Therefore, P_n, the probability that n failed items are in the system and E(B.O.), the expected back orders, may be found as follows. From (5)

$$1 + \sum_{n=1}^{\infty} \Pi_n = 1 + \sum_{n=1}^{A_o} (N\rho)^n + \sum_{n=A_o+1}^{A_o+N} \frac{\rho^n N^{A_o} N!}{(N + A_o - n)!}$$

$$= \sum_{n=0}^{A_o} (N\rho)^n + \sum_{n=A_o+1}^{A_o+N} \frac{\rho^n N^{A_o} N!}{(N + A_o - n)!} .$$

Substituting into (1)

$$P_n = \frac{\Pi_n}{1 + \sum\limits_{n=1}^{\infty} \Pi_n} .$$

Thus we have

$$P_n = \begin{cases} \dfrac{(N\rho)^n}{\sum\limits_{s=0}^{A_o} (N\rho)^s + \sum\limits_{s=A_o+1}^{A_o+N} \dfrac{\rho^s N^{A_o} N!}{(N + A_o - s)!}} , & 0 \leq n \leq A_o \\[3ex] \dfrac{\dfrac{\rho^n N^{A_o} N!}{(N+A_o-n)!}}{\sum\limits_{s=0}^{A_o} (N\rho)^s + \sum\limits_{s=A_o+1}^{A_o+N} \dfrac{\rho^s N^{A_o} N!}{(N + A_o - s)!}} , & A_o < n \leq A_o+N \\[3ex] 0 , & n > A_o+N. \end{cases}$$

In a manner similar to the infinite case, expected back orders, $E(B.O.) = \sum\limits_{n} (B.O.)P_n$, may be found by substituting for

the value of P_n:

$$E(B.O.) = \sum_{n=A_o+1}^{A_o+N} (n-A_o)P_n$$

$$= \sum_{n=A_o+1}^{A_o+N} \left[\frac{\dfrac{(n-A_o)\rho^n N^{A_o} N!}{(N + A_o - n)!}}{\sum_{s=0}^{A_o} (N\rho)^s + \sum_{s=A_o+1}^{A_o+N} \dfrac{\rho^s N^{A_o} N!}{(N + A_o - s)!}} \right].$$

Finally we have

$$E(B.O.) = N^{A_o} N! \sum_{n=A_o+1}^{A_o+N} \left[\frac{(n - A_o)\rho^n}{(N+A_o-n)! \left\{ \sum_{s=0}^{A_o} (N\rho)^s + \sum_{s=A_o+1}^{A_o+N} \dfrac{\rho^s N^{A_o} N!}{(N+A_o-s)!} \right\}} \right] \tag{6}$$

This is the general equation for every advance base.

In the next chapter, we will address the problem of allocating spare items among the advance bases. A marginal approach will be proposed and the two equations of expected back orders which were derived in this chapter will be used in finding optimal decision procedures for allocation.

IV. ALLOCATION OF SPARES

In the preceding chapter, expressions for the expected back orders for the cases of small and large calling populations were derived. It was found that, in both cases, expected back orders depended on the failure rate of items in use (λ), the repair rate at the base's repair facility (μ), and the number of spare items initially stocked at the base (A_o). For the finite source case, expected back orders also depend on the number of items in use on the ships, (N). Although μ, λ and N for any base might in some situations be decision variables, our interest here will be that of allocating a limited number of spares among bases, and thus at each base, A_o will be our decision variable. Appropriate adjusting of A_o for each advance base may change the individual base's and the overall expected back orders. This chapter will propose a marginal approach [2] for minimizing the total of expected back orders over all advance bases, given a limited number of spares to allocate. A numerical example will be shown at the end of the chapter.

From the problem description in Chapter 2, items initially assigned to advance bases can be divided into two kinds: items in use at ships and spare items (some possibly on ships) which are considered as stocked at bases. The total number of items in use over J advance bases is $\sum_{j=1}^{J} N_j$ and the total number of spare items is $\sum_{j=1}^{J} A_{jo}$. The question that arises is, if the Navy has a limited number of

spare items for all advance bases, how many items should be allocated to each of them in order that the overall expected back orders will be minimized. In other words, what is the optimal A_{jo} for the j^{th} base. One approach is to use the marginal expected back orders to determine the allocation.

A. MARGINAL APPROACH

Here the term 'marginal' means the change that occurs by increasing one unit of a variable. The general idea of the marginal approach is to determine the effects on the overall expected back orders when we increase spare items to advance bases, one item at a time. Since the other variables (N,λ,μ) may be different at different advance bases, the marginal expected back orders when a unit is allocated to one base may not be the same as when the unit is allocated to another base. Under the marginal approach, that spare item should be allocated to the advance base where the greatest decrease in total expected back orders is incurred. To use the procedure, all advance bases will first be assumed to have zero spare items. Then items will be allocated by the rule above, one item at a time. In the algorithms that follow, A_o for each base will initially take the value zero. When a spare item has been allocated to an advance base and we are considering allocation of the next spare item, A_o for that base will be incremented one unit.

Expressions for marginal expected back orders are computed as follows:

Case 1

If the number of items in use on the ships at an advance base is very large, the expected back orders for the advance base is as was shown in Equation (4):

$$E(B.O.) = \left(\frac{\lambda}{\mu}\right)^{A_o+1} \left(1 - \frac{\lambda}{\mu}\right)^{-1}.$$

That is, for the j^{th} advance base, given that A_{jo} spare items are allocated, the expected back orders will be

$$E_j(B.O.|A_{jo}) = \left(\frac{\lambda_j}{\mu_j}\right)^{A_{jo}+1} \left(1 - \frac{\lambda_j}{\mu_j}\right)^{-1} . \tag{7}$$

The total expected back orders over all J advance bases will be

$$E_1(B.O.|A_{10}) + E_2(B.O.|A_{20}) + \cdots + E_i(B.O.|A_{io}) + \cdots$$

$$\cdots + E_J(B.O.|A_{J0}).$$

Now, if one more spare item is allocated to the i^{th} advance base, then the total expected back orders will be

$$E_1(B.O.|A_{10}) + E_2(B.O.|A_{20}) + \cdots + E_i(B.O.|A_{io} + 1) +$$

$$+ \cdots + E_J(B.O.|A_{J0}).$$

Therefore, the marginal expected back orders over all advance bases, given one item is allocated to i^{th} base is

$$[E_1(B.O.|A_{10}) + \cdots + E_i(B.O.|A_{io}+1) + \cdots + E_J(B.O.|A_{J0})]$$

$$- [E_1(B.O.|A_{10}) + \cdots + E_i(B.O.|A_{io}) + \cdots + E_J(B.O.|A_{J0})]$$

$$= E_i(B.O.|A_{io} + 1) - E_i(B.O.|A_{io}).$$

Let the marginal quantity be denoted by ΔE_i, so that

$$\Delta E_i = E_i(B.O.|A_{io}+1) - E_i(B.O.|A_{io}). \tag{8}$$

It happens that ΔE_i depends upon expected back orders at the i^{th} base alone; for the following derivation the subscript 'i' will be temporarily dropped. Then

$$\Delta E = \left(\frac{\lambda}{\mu}\right)^{A_o+2}\left(1 - \frac{\lambda}{\mu}\right)^{-1} - \left(\frac{\lambda}{\mu}\right)^{A_o+1}\left(1 - \frac{\lambda}{\mu}\right)^{-1}$$

$$= -\left(\frac{\lambda}{\mu}\right)^{A_o+1}.$$

That is

$$\Delta E_i = -\left(\frac{\lambda_i}{\mu_i}\right)^{A_{io}+1}. \tag{9}$$

Case 2

When N is not large, the marginal expected back orders over all advance bases if one more spare item is assigned to the i^{th} base, ΔE_i, is equal to $E_i(B.O.|A_{io}+1)-E_i(B.O.|A_{io})$. From Chapter 3, the expected back orders for any base is given by Eqn. (6),

$$E(B.O.) = N^{A_o} N! \sum_{n=A_o+1}^{A_o+N} \left[\frac{(n - A_o)\,\rho^n}{(N+A_o-n)!\left\{\sum_{s=0}^{A_o} (N\rho)^s + \sum_{s=A_o+1}^{A_o+N} \frac{\rho^s N^{A_o} N!}{(N+A_o-s)!}\right\}} \right].$$

By dropping the subscript i as before,

$$\Delta E = N^{A_o+1} N! \sum_{n=A_o+2}^{A_o+N+1} \left[\frac{(n - A_o - 1)\, \rho^n}{(N+A_o-n+1)! \left\{ \sum_{s=0}^{A_o+1} (N\rho)^s + \sum_{s=A_o+2}^{A_o+N+1} \frac{\rho^s N^{A_o+1} N!}{(N+A_o-s+1)!} \right\}} \right]$$

$$- N^{A_o} N! \sum_{n=A_o+1}^{A_o+N} \left[\frac{(n - A_o)\, \rho^n}{(N+A_o-n)! \left\{ \sum_{s=0}^{A_o} (N\rho)^s + \sum_{s=A_o+1}^{A_o+N} \frac{\rho^s N^{A_o} n!}{(N+A_o-s)!} \right\}} \right]$$

Let

$$\Pi = \sum_{s=0}^{A_o} (N\rho)^s + \sum_{s=A_o+1}^{A_o+N} \frac{\rho^s N^{A_o} N!}{(N + A_o - s)!} \quad . \tag{10}$$

Then

$$\sum_{s=0}^{A_o+1} (N\rho)^s + \sum_{s=A_o+2}^{A_o+N+1} \frac{\rho^s N^{A_o+1} N!}{(N + A_o - s + 1)!}$$

$$= 1 + \sum_{s=1}^{A_o+1} (N\rho)^s + \sum_{s=A_o+2}^{A_o+N+1} \frac{\rho^s N^{A_o+1} N!}{(N + A_o - s + 1)!}$$

$$= 1 + \sum_{S=0}^{A_o} (N\rho)^{S+1} + \sum_{S=A_o+1}^{A_o+N} \frac{\rho^{S+1} N^{A_o+1} N!}{(N + A_o - S)!}$$

$$= 1 + N\rho\Pi .$$

Since Π does not depend on n, then

$$\Delta E = \frac{N^{A_o+1} N!}{1+N\rho\Pi} \sum_{n=A_o+2}^{A_o+N+1} \frac{(n-A_o+1)\, \rho^n}{(N+A_o-n+1)!} - \frac{N^{A_o} N!}{\Pi} \sum_{n=A_o+1}^{A_o+N} \frac{(n-A_o)\, \rho^n}{(N+A_o-n)!}$$

$$= \frac{N^{A_o+1} N!\, \rho^{A_o+1}}{1 + N\rho\Pi} \sum_{n=1}^{N} \frac{n\,\rho^n}{(N-n)!} - \frac{N^{A_o} N!\, \rho^{A_o}}{\Pi} \sum_{n=1}^{N} \frac{n\,\rho^n}{(N-n)!}$$

$$= \left[\frac{N^{A_o+1} N!\, \rho^{A_o+1}}{1 + N\rho\Pi} - \frac{N^{A_o} N!\, \rho^{A_o}}{\Pi} \right] \sum_{n=1}^{N} \frac{n\,\rho^n}{(N-n)!}$$

$$= - \frac{N^{A_o} N!\, \rho^{A_o}}{\Pi(1 + N\rho\Pi)} \sum_{n=1}^{N} \frac{n\,\rho^n}{(N-n)!} \qquad (11)$$

where Π is as defined in (10).

To demonstrate the approach using the derived equation, a numerical example is given. Consider the allocation of five spare items to three advance bases where the relevant rates are

λ_1 = 2 items/unit of time μ_1 = 3 items/unit of time

λ_2 = 3 items/unit of time μ_2 = 4 items/unit of time

λ_3 = 4 items/unit of time μ_3 = 5 items/unit of time.

If N_1, N_2, N_3 are considerably large, then using (9) will give the result shown in Table I below.

	1st Base		2nd Base		3rd Base		Allocate To
	A_{10}	ΔE	A_{20}	ΔE	A_{30}	ΔE	
1st Spare	0	0.67	0	0.75	0	0.8	3rd Base
2nd Spare	0	0.67	0	0.75	1	0.64	2nd Base
3rd Spare	0	0.67	1	0.56	1	0.64	1st Base
4th Spare	1	0.45	1	0.56	1	0.64	3rd Base
5th Spare	1	0.45	1	0.56	2	0.51	2nd Base

Table I. Optimal Allocation.

That is, allocate one item to the first base and two items each to the second and the third bases.

That is the completion of our work on the allocation problem. The next chapter concludes what has been done in this paper and gives some recommendations for further work on problems of this type.

V. CONCLUSIONS

In previous chapters, we developed models for allocating the spares of a Navy system to it's advance bases. The models are based on the consideration that parts of the Navy system may be viewed as queueing systems. We also developed a procedure for allocation of those spares using a marginal approach. In the last chapter, an example was given to demonstrate the procedure.

The models and procedure derived in this paper are not restricted to a Navy system. Any organization in which subordinates are located large distances apart may face the same problem of allocating spares and can use these models and procedures.

There are some limitations in the work we have done. First, the problem is limited to a single kind of item which is recoverable. Next, each advance base is considered to have only one repair facility. Furthermore, ordering and shipping times between ships and their advance base are considered negligible. Thus there are opportunities for extension and enrichment of this work. Some suggestions are:

1. Considering an organization as a whole, it's subordinates may need different kinds of spare items to replace items in use. When other kinds of items are considered, it may be appropriate to consider them all at the same time since they may be subject to the same set of constraints. For

example, the multi-item problem may be subject to a budget constraint which may create problems such as, given an allowance of budget, determining how many items of each kind are to be acquired as spares for the system, and then how items of each type should be allocated to each subordinate.

2. Each advance base may have more than one repair facility. This creates analytic difficulties since with a multi-server queueing system, it is rather hard to determine the expected back orders. Moreover, there may be central repair facilities to support all the advance bases. This raises the problem to a two-echelon allocation problem, which would be an interesting one to study.

It is hoped that this thesis may be useful to those interested in the problem of allocating limited spares, and that it may be a starting point for studying these problems.

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